Math 132: Differential Topology

§ Winding numbers

Def (Winding number)

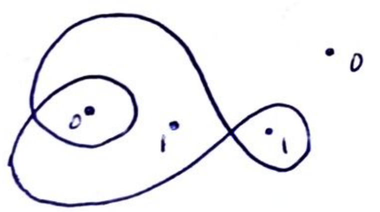
Let M be a compact, ~~connected~~ $(n-1)$-manifold and $f: M \to \mathbb{R}^n$ a smooth map.

If $z \in \mathbb{R}^n$ is not in the image of f, consider the map

$$u : M \longrightarrow S^{n-1}$$
$$x \longmapsto \frac{f(x)-z}{|f(x)-z|}.$$

Define the mod 2 winding number of f around z to be the mod 2 degree of u, i.e.

$$\mathrm{wind}_{\mathbb{Z}/2}(f, z) := \deg_{\mathbb{Z}/2}(u).$$



Thm Suppose $M = \partial D$ for some compact n-manifold D with boundary, and let $F: D \to \mathbb{R}^n$ be a map extending f.

If $z \in \mathbb{R}^n$ is a regular value of F that's not in the image of f, then

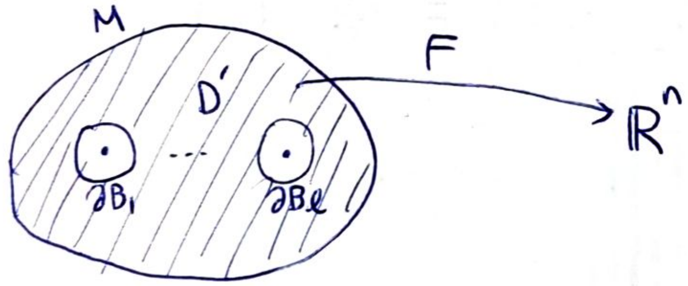
$$\mathrm{wind}_{\mathbb{Z}/2}(f, z) = \# F^{-1}(z) \bmod 2.$$

proof) If z is not in the image of F, then u extends to D, so $\deg_{\mathbb{Z}/2}(u) = 0$ in that case.

If $F^{-1}(z) = \{y_1, \cdots, y_\ell\}$, then F is a local diffeomorphism at each y_i, $1 \le i \le \ell$, so we can choose a small ball B_i around y_i such that $u_i : \partial B_i \to S^{n-1}$ is bijective (and hence $\mathrm{wind}_{\mathbb{Z}/2}(f_i, z) = 1$), where $f_i := F|_{\partial B_i}$. We may assume that the balls are disjoint from each other and from $M = \partial D$.

But our previous observation (applied to $D' = D - \bigcup_{i=1}^{\ell} \mathrm{Int}(B_i)$) implies

$$\mathrm{wind}_{\mathbb{Z}/2}(f, z) = \mathrm{wind}_{\mathbb{Z}/2}(f_1, z) + \cdots + \mathrm{wind}_{\mathbb{Z}/2}(f_\ell, z) \bmod 2$$
$$= \# F^{-1}(z) \bmod 2.$$



$\blacksquare$

When M is a compact, connected hypersurface in $\mathbb{R}^n$, we can use the winding numbers to prove:

codim 1 submanifold

<u>Thm</u> (Jordan-Brouwer separation theorem)
The complement of a compact, connected hypersurface M in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\overline{D}_1$ is a compact n-manifold with boundary $\partial \overline{D}_1 = M$.

proof sketch)

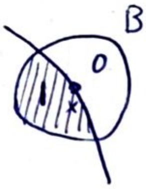
Let $D_a := \{ z \in \mathbb{R}^n \setminus M \mid \mathrm{wind}_{\mathbb{Z}/2}(M, z) = a \}$ for each $a \in \mathbb{Z}/2$,

so that $\mathbb{R}^n \setminus M = D_0 \sqcup D_1$. Since $\mathrm{wind}_{\mathbb{Z}/2}(M,z)$ is locally constant, they are both open.

For each $x \in M$, if B is a small open ball around x in $\mathbb{R}^n$, then

$B \setminus M = B_0 \sqcup B_1$, where B_0, B_1 are two connected half balls,

and $B_a = D_a \cap B$.



← $\mathrm{wind}_{\mathbb{Z}/2}(M,z)$ is locally constant and jumps across M.

Connectivity of M implies D_a is connected, and compactness of M implies $\overline{D_1}$ is compact. ■

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§ Borsuk–Ulam theorem

<u>Thm</u> (Borsuk–Ulam thm)

Let $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be a smooth map such that

$f(-x) = -f(x)$ for all $x \in S^n$, then $\mathrm{wind}_{\mathbb{Z}/2}(f, 0) = 1$.

proof) Let's proceed by induction on n.

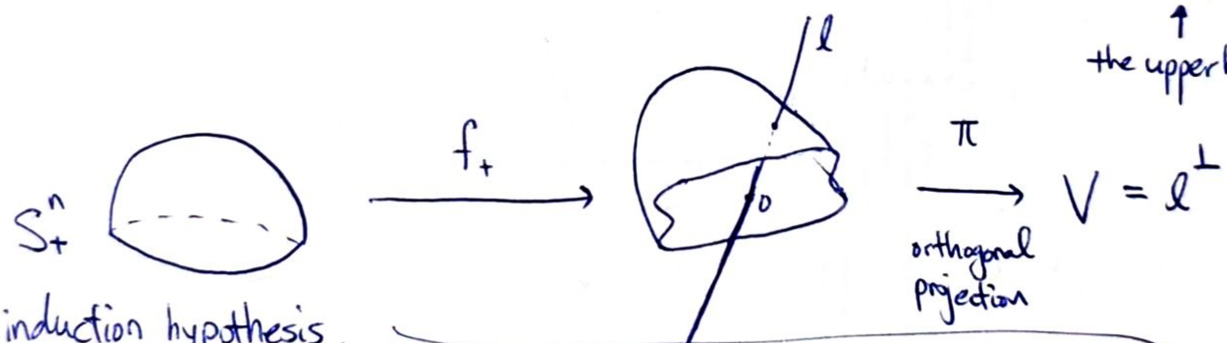
For $n=1$, if $f : S^1 \to S^1$ is an antipodal map, then its lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ must satisfy $\tilde{f}(\theta + \pi) = \tilde{f}(\theta) + k\pi$ for some odd k, and $\deg_{\mathbb{Z}/2} f = k = 1 \pmod 2$

Now assume the thm for $n-1$, and let $f: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be antipodal.
The idea is to compute $\mathrm{wind}_{\mathbb{Z}/2}(f, 0)$ by $I_{\mathbb{Z}/2}(f, R)$ for some ray R from 0.

Let $g = f|_{S^{n-1}}$ ← the equator $S^n \cap \{x_{n+1} = 0\}$, and use Sard to choose
a unit vector v that is a regular value for both
$$\begin{cases} \dfrac{g}{|g|} : S^{n-1} \to S^n, \\[2mm] \dfrac{f}{|f|} : S^n \to S^n. \end{cases}$$

Let $\ell = \mathbb{R} \cdot v$ be the corresponding line. Then $f \pitchfork \ell$, and g never intersects ℓ.

Now,
$$\mathrm{wind}_{\mathbb{Z}/2}(f, 0) = \deg_{\mathbb{Z}/2}\left(\frac{f}{|f|}\right) = \#\left(\frac{f}{|f|}\right)^{-1}(v) \mod 2$$
$$= \frac{1}{2} \# f^{-1}(\ell) = \# f_+^{-1}(\ell) \mod 2, \quad \text{where } f_+ = f|_{S^n_+}.$$
↑ the upper hemisphere



By induction hypothesis,
$$\mathrm{wind}_{\mathbb{Z}/2}(f, 0) = \# f_+^{-1}(\ell) = \# (\pi \circ f_+)^{-1}(0) = \mathrm{wind}_{\mathbb{Z}/2}(\pi \circ g, 0) = 1 \mod 2 \quad \blacksquare$$

<u>Cor</u> If $f: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is antipodal, then f intersects every line through 0 at least once.

<u>Cor</u> For any n smooth functions $g_1, \dots, g_n$ on S^n, there exists a point $p \in S^n$ such that $g_i(p) = g_i(-p)$ for all $i = 1, \dots, n$.

pf) If not, apply the corollary to the map $f(x) = (g_1(x) - g_1(-x), \dots, g_n(x) - g_n(-x), 0)$ and take the x_{n+1} axis for ℓ. $\blacksquare$